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ON METRIZABLE ENVELOPING SEMIGROUPS

BY

Eli Glasner
Department of Mathematics, Tel-Aviv University,
Tel Aviv, Israel
e-mail: glasner@math.tau.ac.il
URL: http://www.math.tau.ac.il/~glasner

AND

Michael Megrelishvili
Department of Mathematics, Bar-Ilan University,
52900 Ramat-Gan, Israel
e-mail: megereli@math.biu.ac.il
URL: http://www.math.biu.ac.il/~megereli

AND

Vladimir V. Uspenskij
Department of Mathematics, 321 Morton Hall, Ohio University,
Athens, Ohio 45701, USA
e-mail: uspensk@math.ohiou.edu

ABSTRACT

When a topological group G acts on a compact space X, its **enveloping semigroup** $E(X)$ is the closure of the set of g-translations, $g \in G$, in the compact space X^X. Assume that X is metrizable. It has recently been shown by the first two authors that the following conditions are equivalent: (1) X is hereditarily almost equicontinuous; (2) X is hereditarily nonsensitive; (3) for any compatible metric d on X the metric $d_G(x,y) := \sup\{d(gx,gy) : g \in G\}$ defines a separable topology on X; (4) the dynamical system (G,X) admits a proper representation on an Asplund Banach space. We prove that these conditions are also equivalent to the following: the enveloping semigroup $E(X)$ is metrizable.

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1. Introduction

A **dynamical system**, or a G-**space**, in this paper is a compact space X ('compact' will mean 'compact and Hausdorff') on which a topological group G acts continuously. We denote such a system by (G, X). For $g \in G$ the g-**translation** (or g-**shift**) is the self-homeomorphism $x \mapsto gx$ of X. If a nonempty subset $Y \subset X$ is G-invariant, i.e. if Y is closed under g-shifts, then Y is a G-**subspace**. The **enveloping semigroup** (or **Ellis semigroup**) of (G, X) is the closure of the set of g-shifts ($g \in G$) in the compact space X^X, equipped with the product topology. Even for simple dynamical systems on a compact metric space the enveloping semigroup may be nonmetrizable. For example, for the classical Bernoulli shift (with $G := \mathbb{Z}$) on the Cantor space $X = \{0, 1\}^{\mathbb{Z}}$, the enveloping semigroup $E(X)$ is homeomorphic to $\beta\mathbb{N}$ (see [18, Exercise 1.25]). If X is the unit interval $[0, 1]$ and $G = H_+[0, 1]$ is the group of all orientation-preserving homeomorphisms, then $E(X)$ is the nonmetrizable space of non-decreasing and end-points-preserving self-maps of $[0, 1]$. If X is a compact manifold without boundary of dimension > 1 and $G = \mathrm{Homeo}\,(X)$ is the group of all self-homeomorphisms of X, then $E(X)$ is X^X.

On the other hand, if G is an equicontinuous group of homeomorphisms of a compact metric space X, then $E(X)$ consists of continuous self-maps of X and hence is metrizable. More generally, the same is true if (G, X) is WAP (= Weakly Almost Periodic). Recall that a function $f \in C(X)$ is **weakly almost periodic** if its G-orbit $\{{}^g f : g \in G\}$ lies in a weakly compact subset of the Banach space $C(X)$, and (G, X) is WAP if every $f \in C(X)$ is WAP. A dynamical system (G, X) is WAP if and only if $E(X)$ consists of continuous self-maps of X [12, 14].

A generalization of WAP systems, called **Radon–Nikodým** (RN for short) systems, was studied in [28, 22]. To define this notion, note that with every Banach space V one can associate a dynamical system $S_V = (H, Y)$ as follows: $H = \mathrm{Iso}\,(V)$ is the group of all linear isometries of V onto itself, equipped with pointwise convergence topology (or the compact-open topology, the two topologies coincide on H), and Y is the unit ball of the dual space V^*, equipped with the weak*-topology. The action of H on Y is defined by $g\phi(v) = \phi(g^{-1}(v))$, $g \in H$, $\phi \in Y$, $v \in V$. The continuity of this action can be easily verified. A **representation** of a dynamical system (G, X) on a Banach space V is a homomorphism of (G, X) to $S_V = (H, Y)$, that is, a pair of continuous maps

(h, α), $h : G \to \mathrm{Iso}\,(V)$ and $\alpha : X \to Y$, such that h is a group homomorphism and $\alpha(gx) = h(g)\alpha(x)$ for all $g \in G$ and $x \in X$. A representation is **proper** if α is a topological embedding.

A compact metric G-space X is WAP if and only if (G, X) admits a proper representation on a reflexive Banach space [28, Corollary 6.10], [22, Theorem 7.6(1)]. A dynamical system is **Radon–Nikodým** (RN) if it admits a proper representation on an Asplund Banach space [28, Definition 3.10], [22, Definition 7.5.2]. (If $G = \{1\}$, we get the class of Radon–Nikodým compact spaces in the sense of Namioka [32].) Recall that a Banach space V is **Asplund** if for every separable subspace $E \subset V$ the dual E^* is separable. An equivalent condition is that the dual Banach space V^* has the Radon-Nikodým property, whence the name RN. Reflexive spaces and spaces of the form $c_0(\Gamma)$ are Asplund. About the history and importance of Asplund spaces see for example [7, 8, 16].

Now assume that X is a metrizable compact space. One of the main results of [22] was a characterization of RN-systems as those which are "close to equicontinuous". To give a precise statement we recall a few definitions from [23, 3, 28, 22].

Let d be a compatible metric on X. We say that (G, X) is **nonsensitive** if for every $\varepsilon > 0$ there exists a nonempty open set $O \subset X$ such that for every $g \in G$ the set gO has d-diameter $< \varepsilon$. (This property does not depend on the choice of a compatible metric d.) A system (G, X) is **hereditarily nonsensitive** (HNS) if all closed G-subsystems are nonsensitive.

A system (G, X) is **equicontinuous at** $p \in X$ if for every $\varepsilon > 0$ there exists a neighborhood O of p such that for every $x \in O$ and every $g \in G$ we have $d(gx, gp) < \varepsilon$. A system is **almost equicontinuous** (AE) if it is equicontinuous at a dense set of points, and **hereditarily almost equicontinuous** (HAE) if every closed subsystem is AE.

Denote by Eq_ε the union of all open sets $O \subset X$ such that for every $g \in G$ the set gO has diameter $< \varepsilon$. Then Eq_ε is open and G-invariant. Let $Eq = \bigcap_{\varepsilon>0} Eq_\varepsilon$. Note that a system (G, X) is nonsensitive if and only if $Eq_\varepsilon \neq \emptyset$ for every $\varepsilon > 0$ and is equicontinuous at $p \in X$ if and only if $p \in Eq$. Suppose that Eq_ε is dense for every $\varepsilon > 0$. Then Eq is dense, in virtue of the Baire category theorem. It follows that (G, X) is AE.

If (G, X) is nonsensitive and $x \in X$ is a **transitive point,** that is, Gx is dense, then for every $\varepsilon > 0$ the open invariant set Eq_ε meets Gx and hence contains Gx. Thus $x \in Eq$. If, in addition, (G, X) is minimal (= all points are transitive), then $Eq = X$. Thus minimal nonsensitive systems are equicontinuous (see [5], [23, Theorem 1.3], [2], or [22, Corollary 5.15]).

THEOREM 1.1 ([22, Theorem 9.14]): *For a compact metric G-space X the following conditions are equivalent:*

(1) *X is RN;*
(2) *X is HNS;*
(3) *X is HAE;*
(4) *every nonempty closed G-subspace Y of X has a point of equicontinuity;*
(5) *for any compatible metric d on X the metric* $d_G(x, y) := \sup_{g \in G} d(gx, gy)$ *defines a separable topology on X.*

It was proved in [22] that the equivalent conditions of Theorem 1.1 imply that the enveloping semigroup $E(X)$ must be of cardinality $\leq 2^\omega$. In fact, it was established in [22, Theorem 14.8] that $E(X)$ is Rosenthal compact (see the first paragraph of Section 6 for a definition), and the question was posed whether this conclusion can be strengthened to "$E(X)$ is metrizable". This question was repeated in [29, Question 7.7]. The aim of the present paper is to answer this question in the affirmative. Moreover, it turns out that metrizablity of $E(X)$, in fact, is equivalent to the conditions of Theorem 1.1:

THEOREM 1.2: *Let X be a compact metric G-space. The following conditions are equivalent:*

(1) *the dynamical system (G, X) is hereditarily almost equicontinuous (HAE);*
(2) *the dynamical system (G, X) is RN, that is, admits a proper representation on an Asplund Banach space;*
(3) *the enveloping semigroup $E(X)$ is metrizable.*

Note that, as the enveloping semigroup depends only on the image of G in Homeo (X), we can deduce that *for metrizable flows the RN property likewise depends only on the image of G in* Homeo (X), *and is therefore independent of the topology of G.* Of course we can obtain these observations also from Theorem 1.1 since HNS (even for non-metrizable flows) has the same property.

After providing a few facts from general topology in Section 2, we prove in Section 3 the implication (2) $\Rightarrow$ (3) of Theorem 1.2, in other words, that for every RN compact metric G-space X the enveloping semigroup $E(X)$ is metrizable. We prove the implication (3) $\Rightarrow$ (1) in Section 4. Since (1) and

(2) are known to be equivalent (Theorem 1.1), this proves Theorem 1.2. The implication (1) $\Rightarrow$ (3) is thus proved via representations on Banach spaces; we give an alternative direct proof in Section 5. Some corollaries of the main theorem are discussed in Section 6. The interested reader is referred to the recent review article on enveloping semigroups in topological dynamics [20].

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2. General topology: prerequisites

A subset of a topological space is **meagre** if it can be covered by a countable family of closed sets with empty interior. A space is **Baire** if every meagre set has empty interior, or, equivalently, if the intersection of any countable family of dense open sets is dense. Let us say that a (not necessarily continuous) function $f : X \to Y$ is **Baire 1** if the inverse image of every open set in Y is F_σ (= the union of countably many closed sets) in X. According to this definition, Baire 1 functions need not be limits of continuous functions. However, if the target space Y is metrizable (or, more generally, perfectly normal), then the limit of every pointwise converging sequence of continuous functions is Baire 1:

Proposition 2.1 (R. Baire): *If Y is a metric space and $\{f_n : X \to Y\}$ is a sequence of continuous functions converging pointwise to $f : X \to Y$ then f is Baire 1.*

Proof. Let $U \subset Y$ be open. There is a sequence $\{F_n\}$ of closed sets such that $U = \bigcup F_n = \bigcup \operatorname{Int} F_n$, where Int denotes the interior. Then $f^{-1}(U)$ is the union over n and k of the closed sets $\bigcap_{i>n} f_i^{-1}(F_k)$. ∎

Proposition 2.2 (R. Baire): *Let $f : X \to Y$ be Baire 1. If X is Baire and Y is separable and metrizable then there exists a dense G_δ-subset A of X such that f is continuous at every $x \in A$.*

Proof. Let $\{U_n : n \in \omega\}$ be a countable base for Y. Write $f^{-1}(U_n) = \bigcup_k F_{nk}$, where each F_{nk} is closed, and consider the union D of the boundaries of all the F_{nk}'s. Then D is meagre, and it is easy to see that f is continuous at every point of the dense G_δ-set $A = X \setminus D$. ∎

PROPOSITION 2.3: *Let $f : X \to Y$ be a (not necessarily continuous) function from a topological space X to a separable metric space Y. Suppose that the inverse image of every closed ball in Y is closed in X. Then f is Baire 1.*

Proof. Every open set U in Y is the union of a countable family of closed balls, hence $f^{-1}(U)$ is F_σ. ∎

We denote by $C(X,Y)$ the space of continuous maps from X to Y, equipped with the compact-open topology. If X is compact and Y is metric, this topology is generated by the sup-metric. If X is compact metrizable then the group $\mathrm{Homeo}\,(X) \subset C(X,X)$ of all self-homeomorphisms of X is a separable and metrizable topological group.

PROPOSITION 2.4: *Let X be Baire, L separable metrizable, K compact metrizable, Y dense in K. If $f : X \to C(K,L)$ is a (not necessarily continuous) function such that for every $y \in Y$ the function $x \mapsto f(x)(y)$ from X to L is continuous, then there exists a dense G_δ-subset A of X such that f is continuous at every $x \in A$.*

The same result is true under the following assumptions: $Y = K$, K is compact but not necessarily metrizable, X is regular and strongly countably complete in the sense of Namioka [31]. For an easier proof of Namioka's theorem that works under less restrictive assumptions, see [33].

Proof. Equip $C = C(K,L)$ with the sup-metric using a compatible metric d on L. Then C is a separable metric space, and the inverse image under f of the closed ball of radius $r > 0$ centered at $h \in C$ is closed, being the intersection of the closed sets $\{x \in X : d(f(x)(y), h(y)) \le r\}$, $y \in Y$. Thus Propositions 2.2 and 2.3 apply. ∎

A function $f : X \to Y$ is **barely continuous** if for every closed nonempty $A \subset X$ the restriction $f|A$ has a point of continuity. (This pun originates in a 1976 paper of E. Michael and I. Namioka, [30].) It is a classical fact (contained in R. Baire's Thesis, 1899) that a function between Polish spaces is barely continuous if and only if it is Baire 1 (see e.g., [26, Theorem 24.15]). If $f : X \to Y$ is an onto barely continuous function between metric spaces and X is separable, then so is Y [30] (see also [22, Lemma 6.5 and Proposition 6.7]). We will need later a G-space version of this statement.

If X and Y are G-spaces, let us say that $f : X \to Y$ is **G-barely continuous** if the restriction $f|A$ has a point of continuity for every G-invariant closed nonempty subset $A \subset X$. A **G-map** between G-spaces is a map commuting with the action of G.

Proposition 2.5: *Let X and Y be metric spaces. Suppose that a (discrete) group G acts on X by homeomorphisms and on Y by isometries. Let $f : X \to Y$ be an onto G-map. If f is G-barely continuous and X is separable, then Y is separable.*

Proof. Pick $\varepsilon > 0$. Let α be the collection of all open subsets U of X such that $f(U)$ can be covered by countably many sets of diameter $\leq \varepsilon$. Then α is G-invariant and closed under countable unions. Since there exists a countable subfamily $\beta \subset \alpha$ such that $\bigcup \beta = \bigcup \alpha$, the family α has a largest element, namely $V = \bigcup \alpha$. Let $A = X \setminus V$. If $a \in A$ is a point of continuity of $f|A$, there exists an open set $O \subset X$ such that $a \in O$ and $f(O \cap A)$ has diameter $\leq \varepsilon$. Then $f(O \cup V) = f(O \cap A) \cup f(V)$ can be covered by countably many sets of diameter $\leq \varepsilon$. Thus $O \cup V \in \alpha$, in contradiction with the fact that O meets the complement of $\bigcup \alpha = V$. We have proved that $f|A$ has no points of continuity. Since A is closed and G-invariant, and f is G-barely continuous, it follows that A is empty.

Thus $X = V \in \alpha$, and Y can be covered by countably many sets of diameter $\leq \varepsilon$. Since ε was arbitrary, Y is separable. ∎

Proposition 2.6: *The Banach dual V^* of a nonseparable Banach space V is nonseparable.*

Proof. Construct a transfinite sequence $\{x_\alpha : \alpha < \omega_1\}$ of unit vectors in V such that for each $\alpha < \omega_1$ the vector x_α does not belong to the closed linear space L_α spanned by the vectors x_β, $\beta < \alpha$. For every $\alpha < \omega_1$ find a functional $f_\alpha \in V^*$ such that $f_a \in L_\alpha^\perp$ and $f_\alpha(x_\alpha) = 1$. All the pairwise distances between distinct f_α's are ≥ 1. It follows that V^*, considered with its norm topology, is not separable. ∎

Proposition 2.7: *Let $f : X \to Y$ be a continuous onto map between compact spaces. If X is metrizable, then so is Y.*

Proof. A compact space K is metrizable if and only if it has a countable base if and only if the Banach space $C(K)$ is separable. Note that $C(Y)$ is isometric to a subspace of $C(X)$ and hence is separable if $C(X)$ is separable.

Alternatively, one can use Arhangelskii's theorem on coincidence of the network weight and weight in compact spaces [15, Theorem 3.1.19]. This approach yields a stronger result: a compact space is metrizable if it is the image under a continuous mapping of any space with a countable base, compact or not. ∎

3. Proof of Theorem 1.2: Part 1

In this section we prove that for every RN compact metric G-space X the enveloping semigroup $E(X)$ is metrizable. Recall that X being RN means that (G, X) has a proper representation on an Asplund Banach space.

For a Banach space V we denote by S_V the dynamical system $(\mathrm{Iso}\,(V), Y)$, where Y is the unit ball of the dual space V^*, equipped with the weak* topology.

We first prove the special case of the implication (2) $\Rightarrow$ (3) of Theorem 1.2, when the dynamical system is of the form S_V, where V is a Banach space with a separable dual.

Proposition 3.1: *Let V be a Banach space with a separable dual, $G = \mathrm{Iso}\,(V)$, Y the compact unit ball of V^* with the weak* topology, considered as a G-space. Then the enveloping semigroup $E(Y)$ is metrizable.*

Proof. Let K be the set of all linear operators of norm ≤ 1 on the Banach space V^*. Consider the topology on K inherited from the product $(V^*)^{V^*}$, where each factor V^* is equipped with the weak* topology. Then K is compact, being a closed subset of the product $\prod_{f \in V^*} \|f\|\, Y$. We claim that K is metrizable. Indeed, V is separable (Proposition 2.6), hence Y is metrizable, and so is each ball rY, $r > 0$. If C is a norm-dense countable subset of V^*, the restriction $A \to A|C$ defines a homeomorphism of K onto a subspace of the product $\prod_{f \in C} \|f\|\, Y$ of countably many metrizable compacta. This proves our claim that K is metrizable.

Restricting each operator $A \in K$ to Y, we obtain a homeomorphism of K with a compact subset L of Y^Y. The enveloping semigroup $E(Y)$ is the closure of the set $\{T^*|Y : T \in G\}$ in L. Since K is metrizable, so are L and $E(Y)$. ∎

Proposition 3.2: *Let G be a separable topological group, X a compact metric G-space. If X is RN then (G,X) has a proper representation on a Banach space with a separable dual.*

Proof. There exists a proper representation $(h,\alpha) : (G,X) \to S_V = (H,Y)$ for some Asplund V. Since $\alpha(X)$ is metrizable, there exists a countable subset $A \subset V$ that separates points of $\alpha(X)$. Let W be the closed linear subspace of V spanned by the union of G-orbits of all points of A. Then W is separable (note that the G-orbit of any point $v \in V$ is separable, being a continuous image of G), G-invariant, and the restriction map $V^* \to W^*$ is one-to-one on $\alpha(X)$. It follows that (G,X) admits a proper representation on W. Since V is Asplund and W is separable, the dual of W is separable. ∎

Proposition 3.3: *Let X be a compact G-space. Suppose that G_1 is a subgroup of G and X_1 is a closed G_1-invariant subset of X. If $E(G,X)$ is metrizable then $E(G_1,X_1)$ also is metrizable.*

Proof. Consider the dynamical systems $D = (G,X)$, $D_1 = (G_1,X)$, and $D_2 = (G_1,X_1)$. The enveloping semigroup $E(D_1)$ is a subspace of $E(D)$, and there is a natural onto map $E(D_1) \to E(D_2)$. If $E(D)$ is metrizable, then so are $E(D_1)$ and $E(D_2)$ (Proposition 2.7). ∎

We now show that for every RN compact metric G-space X the enveloping semigroup $E(G,X)$ is metrizable. Since $E(G,X)$ depends only on the image $\breve{G}$ of G in Homeo(X), we may assume that G is separable. One way to see this is to choose a countable dense subset $\breve{A} \subset \breve{G}$, choose a set of representatives $A = \{a \in G : a \mapsto \breve{a},\ \breve{a} \in \breve{A}\}$, and let G_0 be the countable subgroup of G generated by A. Then clearly (G_0,X) is RN and $E(G,X) = E(G_0,X)$.

By Proposition 3.2 there exists a proper representation

$$(h,\alpha) : (G,X) \to S_V = (H,Y)$$

for some Banach space V with a separable dual. In virtue of Proposition 3.1, the enveloping semigroup of the system $S_V = (H,Y)$ is metrizable. Consider the dynamical system $(h(G),\alpha(X))$. Its enveloping semigroup is metrizable by Proposition 3.3. It remains to note that $E(h(G),\alpha(X))$ and $E(G,X)$ are isomorphic.

4. Proof of Theorem 1.2: Part 2

Let X be a compact metric G-space such that $E(X)$ is metrizable. We prove that X is HAE (= Hereditarily Almost Equicontinuous).

For every closed G-subsystem Y of X the enveloping semigroup $E(Y)$ is metrizable, being a continuous image of $E(X)$. Thus, it suffices to prove that X is AE, that is, that the system (G, X) is equicontinuous at a dense set of points.

Consider the metric space $C = C(E, X)$ of all continuous maps from $E = E(X)$ to X, equipped with the sup-metric. For each $x \in X$ let $x^* \in C$ be the evaluation map defined by $x^*(e) = e(x)$, $e \in E$. It is easy to see that the map $f : X \to C$ defined by $f(x) = x^*$ is continuous at a point $x \in X$ if and only if (G, X) is equicontinuous at x. Thus we must prove that f has a dense set of points of continuity. This follows from Proposition 2.4, where $K = E$, $L = X$ and $Y \subset K$ is the set of all G-translations.

5. An alternative proof of the implication (1) ⇒ (3) in Theorem 1.2

The implication (1) ⇒ (3) in Theorem 1.2: if X is metric and HAE, then $E(X)$ is metrizable — was obtained in an indirect way, via representations on Banach spaces. In this section we give a direct proof in the spirit of Section 4.

Consider the same evaluation map $f : X \to C(E, X)$ as in Section 4. The assumption that X is HAE implies that for every nonempty closed G-invariant subset Y of X the restriction $f|Y$ has a point of continuity. In other words, f is G-barely continuous in the sense of Section 2.

Consider the action of G on E given by $ge(x) = e(g^{-1}x)$ ($g \in G$, $e \in E$, $x \in X$), and the action of G on $C(E, X)$ given by $gh(e) = h(g^{-1}e)$ ($g \in G$, $h \in C(E, X)$, $e \in E$). (We consider here G as a group without topology; these actions need not be continuous if G is considered with its original topology.) Then G acts on $C(E, X)$ by isometries. The evaluation map $f : X \to C(E, X)$ is a G-map. Therefore, we can apply Proposition 2.5: $f(X)$ is a separable subset of $C(E, X)$. Pick a dense countable subset A of $f(X)$. Since $f(X)$ separates points of E, so does A. Therefore, the diagonal product $\triangle A : E \to X^A$ is an embedding. Since X is metrizable and A is countable, X^A is metrizable, and so is E.

6. Some applications and remarks

6.1. Tame dynamical systems. For a topological space X denote by $B_1(X)$ the space of all Baire 1 real-valued functions on X, equipped with the pointwise convergence topology. A compact space K is **Rosenthal** if it is homeomorphic to a subspace of $B_1(X)$ for some Polish X.

In [22, Theorem 3.2] the following dynamical Bourgain-Fremlin-Talagrand dichotomy was established.

Theorem 6.1 (A dynamical BFT dichotomy): *Let (G, X) be a metric dynamical system and let $E(X)$ be its enveloping semigroup. We have the following dichotomy. Either*

(1) *$E(X)$ is separable Rosenthal compact, hence $\operatorname{card} E(X) \leq 2^\omega$; or*
(2) *the compact space E contains a homeomorphic copy of $\beta\mathbb{N}$, hence $\operatorname{card} E(X) = 2^{2^\omega}$.*

In [19] a dynamical system is called **tame** if the first alternative occurs, i.e. $E(X)$ is Rosenthal compact. It is shown in [19] that a minimal metrizable tame system with a commutative acting group is PI. (For the definition of PI and for more details on the structure theory of minimal dynamical systems see e.g., [17].) The authors of three recent works [25], [27] and [21] improve this result to show that under the same conditions the system is in fact an almost 1-1 extension of an equicontinuous system.

Under the stronger assumption that $E(X)$ is metrizable Theorem 1.2 now shows that the commutativity assumption can be dropped and that the system is actually equicontinuous. We get the following definitive result in the spirit of R. Ellis' joint continuity theorem [11].

Theorem 6.2: *A metric minimal system (G, X) is equicontinuous if and only if its enveloping semigroup $E(X)$ is metrizable.*

Proof. It is well-known that the enveloping semigroup of a metric equicontinuous system is a metrizable compact topological group (see e.g., [18, Exercise 1.26]). Conversely, if $E(X)$ is metrizable then, by Theorem 1.2, (G, X) is HAE and being also minimal it is equicontinuous (see the paragraph before Theorem 1.1). ∎

Our characterization of metrizable HNS systems as those having metrizable enveloping semigroups should be compared with the following

Theorem 6.3: *A compact metric dynamical system (G, X) is tame if and only if every element of $E(X)$ is a Baire 1 function from X to itself.*

Proof. If Y is a separable metric space and $B_1(X, Y) \subset Y^X$ is the space of Baire 1 functions from X to Y, then every compact subset of $B_1(X, Y)$ is Rosenthal. Indeed, Y embeds in $\mathbb{R}^{\mathbb{N}}$, hence $B_1(X, Y)$ embeds in $B_1(X, \mathbb{R}^{\mathbb{N}}) = B_1(X \times \mathbb{N})$. In particular, if $E(X) \subset B_1(X, X)$, then $E(X)$ is Rosenthal, which means that (G, X) is tame. Conversely, if $E(X)$ is Rosenthal, then by the Bourgain-Fremlin-Talagrand theorem it is Fréchet [6]. (Recall that a topological space K is **Fréchet** if for every $A \subset K$ and every $x \in \overline{A}$ there exists a sequence of elements of A which converges to x.) In particular, every $p \in E(X) = \overline{G}$ (we may assume that $G \subset \text{Homeo}\,(X)$) is the limit of a sequence of elements of G and therefore of Baire class 1 (Proposition 2.1). ∎

Remarks 6.4: (1) Note that Theorem 1.2 resolves Problem 15.3 in [22]. In fact, since the Glasner–Weiss examples [24] are metric and HNS (see [22, Section 11]) we now know that their enveloping semigroups are metrizable.

(2) Theorem 6.2 answers negatively Problem 3.3 in [19].

(3) In his paper [13] Ellis, following Furstenberg's classical work, investigates the projective action of $GL(n, \mathbb{R})$ on the projective space $\mathbb{P}^{n-1}$. It follows from his results that the corresponding enveloping semigroup is not first countable. In a later work [1], Akin studies the action of $G = GL(n, \mathbb{R})$ on the sphere $\mathbb{S}^{n-1}$ and shows that here the enveloping semigroup is first countable (but not metrizable). The dynamical systems $D_1 = (G, \mathbb{P}^{n-1})$ and $D_2 = (G, \mathbb{S}^{n-1})$ are tame but not RN. Note that $E(D_1)$ is Fréchet, being a continuous image of a first countable space, namely $E(D_2)$.

6.2. Distality and equicontinuity. A dynamical system (G, X) is **distal** if for any two distinct points $x, y \in X$ the closure of the set $\{(gx, gy) : g \in G\}$ in X^2 is disjoint from the diagonal. If X is metrizable and d is a compatible metric on X, this condition means that $\inf_{g \in G} d(gx, gy) > 0$. Every equicontinuous system is distal. By a theorem of Ellis a dynamical system (G, X) is distal if and only if its enveloping semigroup $E(X)$ is (algebraically) a group, see [10]. Note that this characterization implies that for any distal system (G, X) the phase space X is the disjoint union of its minimal subsets. In particular it follows that *a point transitive distal system is minimal.* (A dynamical system (G, X) is **point transitive** if there is some $x \in X$ for which the orbit Gx is dense in X.) As we have already observed, when X is equicontinuous, $E(X)$ is actually a compact topological group.

One version of Ellis' famous joint continuity theorem says that a compact dynamical system (G, X) such that $E(X)$ is a group of continuous maps is necessarily equicontinuous (see [11] and [4, page 60]). Using Ellis's characterizations of WAP and distality this can be reformulated as follows: *A distal WAP system is equicontinuous.* We will now show that the WAP condition can not be much relaxed.

Example 6.5: The following is an example of a dynamical system $(\mathbb{Z}, X)$ which is distal, HAE, and its enveloping semigroup $E(X)$ is a compact topological group isomorphic to the 2-adic integers. However, $(\mathbb{Z}, X)$ is not WAP and a fortiori not equicontinuous.

Let $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ (reals mod 1) be the circle. Let $X = \mathbb{S}\times(\mathbb{N}\cup\{\infty\})$, where $\mathbb{N}\cup\{\infty\}$ is the one point compactification of the natural numbers. Let $T : X \to X$ be defined by:

$$T(s, n) = (s + 2^{-n}, n), \quad T(s, \infty) = (s, \infty).$$

It is not hard to see that $E(X)$ is isomorphic to the compact topological group $\mathbb{Z}_2$ of 2-adic integers. The fact that X is not WAP can be verified directly by observing that $E(X)$ contains discontinuous maps. Indeed, the map $f_a \in E(X)$ corresponding to the 2-adic integer

$$a = \dots 10101 = 1 + 4 + 16 + \cdots$$

can be described as follows: $f_a(s, n) = (s + a_n, n)$, where

$$a_{2k} = \frac{2^{2k} - 1}{3 \cdot 2^{2k}} \to \frac{1}{3}, \quad a_{2k+1} = \frac{2^{2k+2} - 1}{3 \cdot 2^{2k+1}} \to \frac{2}{3}.$$

Geometrically this means that half of the circles are turned by approximately $2\pi/3$, while the other half are turned by approximately the same angle in the opposite direction. The map f_a is discontinuous at the points of the limit circle.

For a point transitive HAE system distality is equivalent to equicontinuity because, as we have seen, a distal point transitive system must be minimal and a minimal HAE system is equicontinuous.

6.3. Semigroup compactifications of groups. A semigroup S is **right topological** if it is equipped with such a topology that for every $y \in S$ the map $x \mapsto xy$ from S to itself is continuous. (Some authors use the term **left**

topological for this.) If for every $y \in S$ the self-maps $x \mapsto xy$ and $x \mapsto yx$ of S both are continuous, S is a **semitopological semigroup**. A **right topological semigroup compactification** of a topological group G is a compact right topological semigroup S together with a continuous semigroup morphism $G \to S$ with a dense range such that the induced action $G \times S \to S$ is continuous. A typical example is the enveloping semigroup $E(X)$ of a dynamical system (G, X) together with the natural map $G \to E(X)$.

Semitopological semigroup compactifications are defined analogously.

We have the following direct corollaries of Theorem 1.2.

COROLLARY 6.6: *For a metric HAE system (G, X) its enveloping semigroup $E(X)$ is again a metrizable HAE system.*

Proof. This follows from Theorem 1.2 because the enveloping semigroup of the flow $(G, E(G, X))$ is isomorphic to $E(G, X)$. ∎

COROLLARY 6.7: *The following three classes coincide:*

(1) Metrizable enveloping semigroups of G-systems.
(2) Enveloping semigroups of HAE metrizable G-systems.
(3) Metrizable right topological semigroup compactifications of G.

Proof. A dynamical system has the structure of a right topological semigroup compactification of G if and only if it is the enveloping semigroup of some dynamical system (see e.g., [18, Section 1.4] or [22, Section 2]). ∎

REMARK 6.8: It is well-known that the enveloping semigroup of a WAP dynamical system is a semitopological semigroup compactification of G (see e.g., [18, Section 1.4] or [22, Section 2]). Thus a WAP version of Corollary 6.7 (omitting part (1)) can be obtained by changing 'HAE' to 'WAP' and 'right topological semigroup' to 'semitopological semigroup'. Moreover, as was shown in [9] (see also [18, Theorem 1.48]), when the acting group G is commutative, a point transitive WAP system is always isomorphic to its enveloping semigroup, which in this case is a commutative semitopological semigroup. Thus, for such G the class of all metric, point transitive, WAP systems coincides with that of all metrizable, commutative, semitopological semigroup compactifications of G.

6.4. SEMIGROUP ACTIONS. Our main result (Theorem 1.2) remains true for semigroup actions up to a more flexible version of HAE. Namely, we say that a continuous action of a topological semigroup S on a metric space (X, d) is HAE if for every (not necessarily S-invariant) closed nonempty subset Y there exists a dense subset $Y_0 \subset Y$ such that every point $y_0 \in Y_0$ is a point of continuity of the natural inclusion map $(Y, d|_Y) \to (X, d_S)$, where $d_S(x, y) := \sup_{s \in S} d(sx, sy)$. (It is not hard to see that for G-group actions on compact metric spaces this definition is equivalent to our old definition which involved only G-invariant closed subsets.) Then again HAE, RN and the metrizability of $E(X)$ are equivalent. We omit the details.

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